Some Undecidable Approximations of TRSs

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Abstract In this paper we study the decidability of reachability, normalisation, and neededness in n-shallow and n-growing TRSs. In an n-growing TRS, a variable that occurs on both the left-hand and right-hand side of a rewrite must be at depth n on the left-hand side an at a depth greater than n on the right-hand side. In an n-shallow TRS, a variable that occurs on both the left-hand and right-hand side of a rewrite rule must be at depth n on both sides.

The concepts n-growing and n-shallow are generalisations of the concepts growing and shallow, as introduced respectively by Jacquemard and Comon. For both shallow and growing TRSs reachability, normalisation, and neededness are decidable. However, as we show, these results do not generalise to n-growing and n-shallow TRSs. Consequently, a needed reduction strategy cannot be effectively employed in n-growing or n-shallow TRS.

1 Introduction

As is well-known, given an arbitrary term rewriting system (TRS), the following questions are undecidable.

- Reachability: is a term reachable from another term [4]?
- Normalisation: does a term have a normal form [1]?
- Neededness: does a term have a needed redex [4]?

However, for some classes of TRSs these properties are decidable. These classes are often used as approximations. That is, let $\mathcal{R}$ and $\mathcal{S}$ be TRSs over the same signature, then $\mathcal{S}$ is an *approximation* of $\mathcal{R}$ if $\to_{\mathcal{R}}^{*} \subseteq \to_{\mathcal{S}}^{*}$ and $\mathsf{NF}_{\mathcal{R}} = \mathsf{NF}_{\mathcal{S}}$. Here, $\to_{\mathcal{R}}^{*}$ and $\to_{\mathcal{S}}^{*}$ denote the rewrite relations of $\mathcal{R}$ and $\mathcal{S}$, and $\mathsf{NF}_{\mathcal{R}}$ and $\mathsf{NF}_{\mathcal{S}}$ denote the sets of normal forms of $\mathcal{R}$ and $\mathcal{S}$.

Two remarks are in order with respect to classes for which reachability, normalisation, and neededness are decidable. First, in most of these classes the shapes of the rewrite rules are restricted. See, for example, [4,2,5,9]. Second, given the decidability of neededness a needed reduction strategy can effectively be employed. See, for example, [4,3,2,5].

In this paper we explore the boundaries of the decidability of reachability, normalisation, and neededness. We do this by introducing n-growing and n-shallow TRSs. These TRSs are generalisations of the growing and shallow TRSs as introduced respectively by Jacquemard [5] and Comon [2]. Although reachability, normalisation, and neededness are decidable for growing and shallow TRSs we show that these properties are undecidable for our generalisations.

The n-growing and n-shallow TRSs are also closely related to four other classes of TRSs for which reachability, normalisation, and neededness are undecidable [5,6,7]. We show that n-growing and n-shallow TRSs are different from those classes of TRSs except in one instance.

We proceed as follows. In Sect. 2 we give some preliminary definitions. Then, in Sect. 3 we define two variants of Post's Correspondence Problem (PCP). We use these variants in Sect. 4 and Sect. 5 to show that reachability, normalisation, and neededness are undecidable for n-growing and n-shallow TRSs. In Sect. 6 we compare the n-growing and n-shallow TRSs to the other four classes of TRSs for which reachability, normalisation, and neededness are undecidable. In the final section, Sect. 7, we give some directions for further research.

2 Preliminaries

Throughout this paper we assume Γ is an arbitrary alphabet. By Γ^* and Γ^+ we denote the set of finite strings and the set finite non-empty strings over Γ. Moreover, by ϵ we denote the empty string, and if $s \in \Gamma^*$, then $|s|$ denotes the length of s.

If $s, t \in \Gamma^*$, then $s \cdot t$ denotes the concatenation of s and t. The empty string ϵ is the neutral element for concatenation. If $a \in \Gamma$ and $n \in \mathbb{N}$, then $a^0 = \epsilon$ and $a^{n+1} = a \cdot a^n$.

By $\mathcal{T}er(\Sigma, X)$ we denote the set of *terms* over the signature Σ and the set of variables X. The set Σ_n denotes the subset of Σ whose elements have arity n. If $t \in \mathcal{T}er(\Sigma, X)$, then $\mathcal{V}ar(t)$ denotes the set that contains the variables that occur in t We call t *linear* if each variable occurs at most once in t.

We confuse signatures consisting only of unary function symbols and alphabets. Hence, given a unary function symbol f and an $n \in \mathbb{N}$ we have $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$.

We denote the set of positions of a term $t \in \mathcal{T}er(\Sigma, X)$ by $\mathcal{P}os(t) \subseteq \mathbb{N}^*$. The *depth* of a subterm at $p \in \mathcal{P}os(t)$ is $|p|$. When t is linear, each $x \in \mathcal{V}ar(t)$ has a unique depth. In that case, $d_t(x)$ denotes the depth of x in t.

By $\mathcal{R} = (\Sigma, R)$ we denote a term rewriting system (TRS) with the signature Σ and the set of rewrite rules R. The elements of R are denoted $l \to r$. As usual in the study of approximations we only require $l \notin X$. We do not require not $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The transitive reflexive closure of $\to$ is denoted by $\to^*$. The rule $l \to r$ is called linear if l and r are linear.

Let $\mathcal{R} = (\Sigma, R)$ be a TRS and $s, t \in \mathcal{T}er(\Sigma, X)$. The term t is *reachable* from s if $s \to^* t$. Moreover, s *normalises* if $s \to^* t$ and if t is a normal form. A redex in s is *needed* if a descendant of the redex is contracted in every reduction from s to a normal form. A *needed reduction strategy* contracts in every term an arbitrary needed redex.

3 Variants of Post's Correspondence Problem

In this section we introduce two variants of Post's Correspondence Problem (PCP). In the definition of the variants we use the following notation.

Definition 1. Let $s \in \Gamma^+$. Given an $n \in \mathbb{N}$, define $s^{[n]}$ by

- $s^{[n]} = a^n$ if $s = a$ with $a \in \Gamma$, and
- $s^{[n]} = a^n \cdot t^{[n]}$ if $s = a \cdot t$ with $a \in \Gamma$ and $t \in \Gamma^+$.

Note that this is not exponentiation. That is defined as $s^0 = \epsilon$ and $s^{n+1} = s \cdot s^n$. Assuming $\Gamma = \{a, b\}$, we have $(ab)^{[2]} = aabb$ and $(ab)^2 = abab$.

We now define three kinds of pairs. We use them in the definition of PCP and the two PCP variants.

Definition 2. Let $u, v \in \Gamma^+$ and n a natural number. Then, (u, v) is called a *PCP pair*, $(u^{[n]}, v^{[n]})$ is called an *n-PCP pair*, and $(u^{[n]} \cdot e^{[kn]}, v^{[n]} \cdot e^{[ln]})$ is called a *padded n-PCP pair* if $k = \max\{|u|, |v|\} - |u|$, $l = \max\{|u|, |v|\} - |v|$, and $e \notin \Gamma$.

The intuition behind a padded n-PCP pair is that it is an n-PCP pair in which the shortest string is padded with e symbols. This gives both strings in the pair the same length. We have for any padded n-PCP pair $(u^n \cdot e^{kn}, v^n \cdot e^{ln})$ that

$$n(|u| + k) = |u^{[n]} \cdot e^{[kn]}| = |v^{[n]} \cdot e^{[ln]}| = n(|v| + l)$$

and that

$$((u^{[n]} \cdot e^{[kn]})[e := \epsilon], (v^{[n]} \cdot e^{[ln]})[e := \epsilon]) = (u^{[n]}, v^{[n]}) .$$

We now define PCP and the two PCP variants.

Question 1 (PCP). Let P be a finite set of PCP pairs. Does there exist an $m \geq 1$ such that $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_m$ with $(u_i, v_i) \in P$ for all $1 \leq i \leq m$?

Question 2 (n-PCP). Let P be a finite set of n-PCP pairs. Does there exist an $m \geq 1$ such that $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_m$ with $(u_i, v_i) \in P$ for all $1 \leq i \leq m$?

Question 3 (Padded n-PCP). Let $e \notin \Gamma$ and let P be a finite set of padded n-PCP pairs. Does there exist an $m \geq 1$ such that $(u_1 \cdot \ldots \cdot u_m)[e := \epsilon] = (v_1 \cdot \ldots \cdot v_m)[e := \epsilon]$ with $(u_i, v_i) \in P$ for all $1 \leq i \leq m$?

The three questions can be transformed into each other, as there exists for each kind of pair a related pair of each of the other kinds. For example, assuming again $\Gamma = \{a, b\}$, we have for the PCP pair (a, ab) that $(a^{[n]}, (ab)^{[n]})$ is a related n-PCP pair and that $((ae)^{[n]}, (ab)^{[n]})$ is related padded n-PCP pair. This leads to the following theorem.

Theorem 1. *PCP, n-PCP, and padded n-PCP are reducible to each other for* $n \geq 1$.

Proof. Using the previously described relations between the pairs, this follows directly from the definitions of PCP, n-PCP and padded n-PCP. □

By the previous theorem and the undecidability of PCP [8], we have the following.

Corollary 1. *The n-PCP and padded n-PCP questions are undecidable for* $n \geq 1$.

For padded n-PCP note that if $(u_1 \cdot \ldots \cdot u_m)[e := \epsilon] = (v_1 \cdot \ldots \cdot v_m)[e := \epsilon]$, then the number of e occurrences must be equal in $u_1 \cdot \ldots \cdot u_m$ and $v_1 \cdot \ldots \cdot v_m$. If not, the lengths of $(u_1 \cdot \ldots \cdot u_m)[e := \epsilon]$ and $(v_1 \cdot \ldots \cdot v_m)[e := \epsilon]$ are different which contradicts equality.

Using the previous fact and assuming a string rewrite system (SRS) with for all $a \in \Gamma$ and $e \notin \Gamma$ the rewrite rules $a \cdot e \to e \cdot a$ and $e \cdot a \to a \cdot e$, we can rephrase padded n-PCP.

Question 4 (Padded n-PCP). Let $\Delta = \Gamma \cup \{e\}$ and let P be a finite set of padded n-PCP pairs. Does there exist an $m \geq 1$ and an $s \in \Delta^+$ such that $u_1 \cdot \ldots \cdot u_m \to^* s$ and $v_1 \cdot \ldots \cdot v_m \to^* s$ with $(u_i, v_i) \in P$ for all $1 \leq i \leq m$?

4 Undecidable n-Growing Term Rewriting Systems

In this section we describe our first class of TRSs for which reachability, normalisation, and neededness are undecidable. The class is defined as follows.

Definition 3. Let $l \to r$ be a rewrite rule. The rule is *n-growing* if it is linear and if for all $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ it holds that $d_l(x) = n$ and $d_r(x) > n$. A TRS is *n-growing* if all its rewrite rules are n-growing.

Observe that in n-growing TRSs we restrict the shapes of the rewrite rules. Moreover, observe that n-growing rewrite rules and TRSs are closely related to the following rewrite rules and TRSs, as defined by Jacquemard [5].

Definition 4. Let $l \to r$ be a rewrite rule. The rule is *growing* if it is linear and if for all $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ it holds that $d_l(x) = 1$. A TRS is *growing* if all its rewrite rules are growing.

Obviously, for $n = 1$ the n-growing TRSs form a sub-class of the growing TRSs. For $n > 1$ the n-growing TRSs do not form a sub-class. We have, for example, the n-growing rewrite rule

$$f^n(x) \to f^{n+1}(x) \, .$$

For $n > 1$ this rewrite rule is not growing, as $d_{f^n(x)}(x) = n > 1$.

The growing TRSs do not form a sub-class of the n-growing TRSs for any n. We have, for example, the growing rewrite rule

$$f(x) \to x \, .$$

This rewrite rule is not n-growing, as $d_x(x) = 0$.

Using tree automata techniques, Jacquemard [2] proves that reachability and normalisation are decidable for growing TRSs. Durand and Middeldorp [3] prove that neededness is decidable for growing TRSs. As each 1-growing TRS is growing, we also have decidability of reachability, normalisation, and neededness for 1-growing TRSs. However, as we show next, these results do not generalise to n-growing TRSs with $n > 1$.

Theorem 2. *Let $n \geq 1$. Reachability is undecidable for $n+1$-growing TRSs.*

Proof. We reduce n-PCP to reachability in an $n+1$-growing TRS. Suppose we have a finite set P of n-PCP pairs. Define the signature $\Sigma = \Gamma \uplus \{c, d, f, g, h\}$, where $\uplus$ denotes the disjoint union. The arities are as follows

- c and d are constants,
- g, h and the elements of Γ have arity 1, and
- f has arity 2.

Also define for all $(u, v) \in P$ and $a \in \Gamma$ the following rewrite rules

$$\begin{aligned} c &\to f(g^n(u(d)), g^n(v(d))) && (1) \\ f(g^n(x), g^n(y)) &\to f(g^n(u(x)), g^n(v(y))) && (2) \\ f(g^n(x), g^n(y)) &\to h^{n+1}(f(x, y)) && (3) \\ f(a^n(x), a^n(y)) &\to h^{n+1}(f(x, y)) && (4) \\ f(d, d) &\to d && (5) \\ h^{n+1}(d) &\to d && (6) \end{aligned}$$

As is easy to see, we have a finite number of n-growing rewrite rules. Hence, the rewrite rules form an n-growing TRS. By $n \geq 1$, we have for the TRS that d is reachable from c if and only if n-PCP has a solution for P.

To see that d is reachable from c if n-PCP has a solution for P, suppose that $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_m$ is a solution. We can now construct the reduction sequence

$$\begin{aligned} c &\to_{(1)} f(g^n(u_m(d)), g^n(v_m(d))) \\ &\to^*_{(2)} f(g^n(u_1 \cdot \ldots \cdot u_m(d)), g^n(v_1 \cdot \ldots \cdot v_m(d))) \\ &\to_{(3)} h^{n+1}(f(u_1 \cdot \ldots \cdot u_m(d), v_1 \cdot \ldots \cdot v_m(d))) \end{aligned}$$

As $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_m$, we can, for some k, extend the reduction sequence to

$$\begin{aligned} c &\to^* \ h^{n+1}(f(u_1 \cdot \ldots \cdot u_m(d), v_1 \cdot \ldots \cdot v_m(d))) \\ &\to^*_{(4)} h^{k(n+1)}(f(d,d)) \\ &\to^*_{(5)} f(d,d) \\ &\to_{(6)} d \end{aligned}$$

Hence, d is reachable from c.

To see that n-PCP has a solution for P if d is reachable from c, note that the only way to reduce c to d is to first perform an (1)-step and a number of (2)-steps, then to perform a (3)-step and a number of (4)-steps, and to finally perform a (5)-step and a number of (6)-steps. Also note that the reduction sequence only ends in d if the (1)-step and the (2)-steps give us a term $f(g^n(u_1 \cdot \ldots \cdot u_m(d)), g^n(v_1 \cdot \ldots \cdot v_m(d)))$ with $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_m$. That is, as n-PCP has a solution for P.

Thus, n-PCP is reducible to a reachability problem in an $n+1$-growing TRS. As n-PCP is undecidable for $n \geq 1$, so is reachability for $n+1$-growing TRSs with $n \geq 1$. □

Observe that if we assume $n = 0$ in the previous proof, the rewrite rule

$$f(a^n(x), a^n(y)) \to h^{n+1}(f(x,y))$$

collapses to

$$f(x,y) \to h^{n+1}(f(x,y))\,.$$

As a consequence, d is no longer reachable form c. Something like this was to be expected, as reachability is decidable for 1-growing TRSs.

We now extend the above result to normalisation and neededness.

Theorem 3. *Let $n \geq 1$. Normalisation is undecidable for $n+1$-growing TRSs.*

Proof. We reduce n-PCP to normalisation in an $n+1$-growing TRS employing the proof showing that n-PCP is reducible to reachability.

Note that adding the $n+1$-growing rewrite rule

$$h^{n+1}(x) \to h^{n+1}(h^{n+1}(x)) \tag{7}$$

to the TRS from the proof of Theorem 2 does not change the fact that d is reachable from c if and only if n-PCP has a solution for P. However, by adding the rule, if d is reachable from c, the term d becomes the only normal form of c, else c does not have a normal form. By this fact and the fact that d is reachable from c if and only if n-PCP has a solution for P, we have that c has a normal form if and only if n-PCP has a solution for P.

Thus, n-PCP is reducible to normalisation in an $n+1$-growing TRS. As n-PCP is undecidable for $n \geq 1$, so is normalisation for $n+1$-growing TRSs with $n \geq 1$. □

Theorem 4. *Let $n \geq 1$. Neededness is undecidable for $n+1$-growing TRSs.*

Proof. We reduce n-PCP to neededness in an $n+1$-growing TRS employing the proof showing that n-PCP is reducible to normalisation.

By the proof of Theorem 3 only d can be a normal form of c. Hence, the only redex in c is needed if and only if d actually is a normal form of c. By this fact and the fact that c normalises if and only if n-PCP has a solution for P, we have that c has a needed redex if and only if n-PCP has a solution for P.

Thus, n-PCP is reducible to neededness in an $n+1$-growing TRS. As n-PCP is undecidable for $n \geq 1$, so is neededness for $n+1$-growing TRSs with $n \geq 1$. □

As a consequence of the previous theorem we have that a needed reduction strategy cannot be effectively employed in an $n+1$-growing TRS with $n \geq 1$.

5 Undecidable n-Shallow Term Rewriting Systems

In this section we describe our second class of TRSs for which reachability, normalisation, and neededness are undecidable. The class is defined as follows.

Definition 5. Let $l \to r$ be a rewrite rule. The rule is *n-shallow* if it is linear and if for all $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ it holds that $d_l(x) = d_r(x) = n$. A TRS is *n-shallow* if all its rewrite rules are n-shallow.

Observe that, like n-growing TRSs, n-shallow TRSs have restrictions on the shapes of their rewrite rules. The n-shallow TRSs form neither a sub-class nor a super-class of the n-growing TRSs. Consider, for example, the n-shallow rewrite rule

$$f^n(x) \to f^n(x)\,.$$

This rewrite rule is not n-growing, as $d_{f^n(x)}(x) = d_{f^n(x)}(x)$. We also have the n-growing rewrite rule

$$f^n(x) \to f^{n+1}(x)\,.$$

This rewrite rule is not n-shallow, as $d_{f^n(x)}(x) < d_{f^{n+1}(x)}(x)$.

The n-shallow rewrite rules and TRSs are closely related to the following rewrite rules and TRSs, as defined by Comon [2].

Definition 6. Let $l \to r$ be a rewrite rule. The rule is *shallow* if it is linear and if for all $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ it holds that $d_l(x) = 1$ and $d_r(x) \leq 1$. A TRS is *shallow* if all its rewrite rules are shallow.

Obviously, for $n = 1$ the n-shallow TRSs form a sub-class of the shallow TRSs. For $n > 1$ the n-shallow TRSs do not form a sub-class. For example, consider again the n-shallow rewrite rule

$$f^n(x) \to f^n(x)\,.$$

For $n > 1$ this rewrite rule is not shallow, as $d_{f^n(x)}(x) = d_{f^n(x)}(x) = n > 1$.

The shallow TRSs do not form a sub-class of the n-shallow TRSs for any n. This follows by the same example that shows that the growing TRSs do not form a sub-class of the n-growing TRSs for any n.

Using tree automata techniques, Comon [2] proves that reachability and normalisation are decidable for shallow TRSs. Durand and Middeldorp [3] prove that neededness is decidable for shallow TRSs. As each 1-shallow TRS is growing, we also have decidability of reachability, normalisation, and neededness for 1-shallow TRSs. However, as we show next, these results do not generalise to n-shallow TRSs with $n > 1$.

In the proofs below, we denote $[a, b]$ with $a, b \in \Delta = \Gamma \cup \{e\}$ and $e \notin \Gamma$ a unary function symbol. We also use the following definition.

Definition 7. Let $u, v \in \Delta^+$ with $|u| = |v|$. For $a, b \in \Delta$ and $u', v' \in \Delta^+$, define $[u, v](x)$ by

- $[u, v](x) = [a, b](x)$ if $u = a$ and $v = b$, and
- $[u, v](x) = [a, b]([u', v'](x))$ if $u = a \cdot u'$ and $v = b \cdot v'$.

Theorem 5. *Let $n \geq 1$. Reachability is undecidable for $n + 1$-shallow TRSs.*

Proof. We reduce padded $n+1$-PCP to reachability in an $n+1$-shallow TRS. Suppose we have a finite set P of padded $n+1$-PCP pairs and an $e \notin \Gamma$. Let $\Delta = \Gamma \cup \{e\}$. Define the signature $\Sigma = \{[a, b] \mid a, b \in \Delta\} \uplus \{c, d\}$, with each $[a, b]$ a unary function symbol and c, d constants. Also define for all $(u, v) \in P$ and $a \in \Delta$ the following rewrite rules

$$c \to [u, v](c) \tag{1}$$

$$[a, a](c) \to d \tag{2}$$

$$[a, a](d) \to d \tag{3}$$

Moreover, define for all $u, u', v, v' \in \Delta^+$ with $u[e := \epsilon] = u'[e := \epsilon]$, $v[e := \epsilon] = v'[e := \epsilon]$, and $|u| = |u'| = |v| = |v'| = n + 1$ the following rewrite rule

$$[u, v](x) \to [u', v'](x) \tag{4}$$

This last rewrite rule can be considered as a transitive application to the strings u and v of the rewrite rules given just before Question 4.

As is easy to see, we have a finite number of n-shallow rewrite rules. Hence, the rewrite rules form an $n + 1$-shallow TRS. By $n \geq 1$, we have for the TRS that d is reachable from c if and only if padded $n + 1$-PCP has a solution for P.

To see that d is reachable from c if padded n-PCP has a solution for P, suppose that for some $s \in \Delta^+$ we have $u_1 \cdot \ldots \cdot u_m \to^* s$ and $v_1 \cdot \ldots \cdot v_m \to^* s$. That is, padded n-PCP has a solution for P. We can construct the following reduction sequence

$$\begin{aligned} c &\to_{(1)} [u_1, v_1](c) \\ &\to^*_{(1)} [u_1 \cdot \ldots \cdot u_m, v_1 \cdot \ldots \cdot v_m](c) \end{aligned}$$

As $u_1 \cdot \ldots \cdot u_m \to^* s$ and $v_1 \cdot \ldots \cdot v_m \to^* s$, we can, for some k, extend the reduction sequence to

$$\begin{aligned} c &\to^* \ [u_1 \cdot \ldots \cdot u_m, v_1 \cdot \ldots \cdot v_m][c] \\ &\to^*_{(4)} [s, s](c) \\ &\to_{(2)} \cdots \\ &\to^*_{(3)} d \end{aligned}$$

Hence, d is reachable from c.

To see that n-PCP has a solution for P if d is reachable from c, note that the only way to reduce c to d is to first perform a number of (1)-steps, then to perform a (2)-step, and to finally perform a number of (3)-steps and to interleave all these steps with (4)-steps. Also note that the reduction sequence only ends in d if the (1)-steps together with a number of (4)-steps give us a term $[s, s](c)$ for some $u_1 \cdot \ldots \cdot u_m \to^* s$ and $v_1 \cdot \ldots \cdot v_m \to^* s$. That is, as padded n-PCP has a solution for P.

Thus, padded $n+1$-PCP is reducible to a reachability problem in an $n+1$-shallow TRS. As padded $n+1$-PCP is undecidable for $n \geq 1$, so is reachability for $n+1$-shallow TRSs with $n \geq 1$. □

Observe that if we assume $n = 0$ in the previous proof, the last rewrite rule collapses to

$$[a, b](x) \to [a, b](x)$$

with $a, b \in \Delta$. As a consequence, d is no longer reachable from c. Something like this was to be expected, as reachability is decidable for 1-shallow TRSs.

Note that by the TRS specified in the previous proof an even stronger property holds.

Theorem 6. *Let $n \geq 1$. Reachability is undecidable for $n+1$-shallow TRSs in which every rewrite rule has at most one variable which occurs both at the left-hand and right-hand side of the rewrite rule.*

We now extend the above result to normalisation and neededness.

Theorem 7. *Let $n \geq 1$. Normalisation is undecidable for $n+1$-shallow TRSs.*

Proof. We reduce padded $n+1$-PCP to normalisation in an $n+1$-shallow TRS employing the proof showing that n-PCP is reducible to reachability.

Note that adding for all $u, v \in \Delta^+$ with $|u| = |v| \leq n+1$ the $n+1$-shallow rewrite rule

$$[u, v](d) \to [u, v](d) \tag{5}$$

to the TRS from the proof of Theorem 5 does not change the fact that d is reachable from c if and only if padded n-PCP has a solution for P. However, by adding the rule, if d is reachable for c, the term d becomes the only normal form of c, else c does not have a normal form. By this fact and the fact that d is reachable from c if and only if padded $n+1$-PCP has a solution for P, we have that c has a normal form if and only if padded $n+1$-PCP has a solution for P.

Thus, padded $n+1$-PCP is reducible to normalisation in an $n+1$-shallow TRS. As padded $n+1$-PCP is undecidable for $n \geq 1$, so is normalisation for $n+1$-shallow TRSs with $n \geq 1$. □

Theorem 8. *Let $n \geq 1$. Neededness is undecidable for $n+1$-shallow TRSs.*

Proof. We reduce padded $n+1$-PCP to neededness in an $n+1$-shallow TRS. We employ the proof showing that padded $n+1$-PCP is reducible to normalisation.

By the proof of Theorem 7 only d can be a normal form of c. Hence, the only redex in c is needed if and only if d actually is a normal form of c. By this fact and the fact that c normalises if and only if padded $n+1$-PCP has a solution for P, we have that c has a needed redex if and only if padded $n+1$-PCP has a solution for P.

Thus, padded $n+1$-PCP is reducible to neededness in an $n+1$-shallow TRS. As padded $n+1$-PCP is undecidable for $n \geq 1$, so is neededness for $n+1$-shallow TRSs with $n \geq 1$. □

As a consequence of the previous theorem we have that the needed reduction strategy cannot be effectively employed in an $n+1$-shallow TRS with $n \geq 1$.

6 Related Work

In this section, we compare n-growing and n-shallow TRSs to four other classes of TRSs for which reachability, normalisation, and neededness are undecidable [5,6,7]. We use the following five rewrite rules in the comparison.

$$f(x,x) \to g(x) \tag{1}$$
$$f(f(x)) \to x \tag{2}$$
$$f(g(x)) \to c \tag{3}$$
$$f^n(x) \to f^{n+1}(x) \tag{4}$$
$$f^n(x) \to f^n(x) \tag{5}$$

Note that (1) and (2) are neither n-growing nor n-shallow, that (3) is n-growing and n-shallow, and that (4) and (5) are respectively n-growing and n-shallow.

The first class of TRSs, defined by Jacquemard [5], requires for all $l \to r$ and $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ that $d_l(x) \leq 1$. This definition is equal to the definition of growing, except that rewrite rules no longer need to be linear. Hence, 1-growing and 1-shallow TRSs form sub-classes. For $n > 1$ the n-growing and n-shallow TRSs do not form sub-classes by (4) and (5). Moreover, for all n the n-growing and n-shallow TRSs do not form super-classes by (1), which is not n-growing or n-shallow as the left-hand side is not linear.

The second class of TRSs, again defined by Jacquemard [5], requires for all $l \to r$ and $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ that either $d_l(x) \leq 1$ or $d_r(x) \leq 1$ and that $l \to r$ is linear. Again, this definition is equal to the definition of growing, except that in this case $d_r(x) \leq 1$ is allowed. Consequently, 1-growing and 1-shallow TRSs again form sub-classes. For $n > 1$ the n-growing and n-shallow TRSs do not form sub-classes by (4) and (5). Moreover, for all n the n-growing and n-shallow TRSs do not form super-classes by (2), which is not n-growing or n-shallow as $d_x(x) = 0$.

The third class of TRSs, also defined by Jacquemard [6], requires for all $l \to r$ that l and r are of the shape $f(t_1, \ldots, t_k)$ with $f \in \Sigma \cup X$ and with t_i either a variable or a ground term for all $1 \leq i \leq k$. For all n the n-growing and n-shallow TRSs do not form sub-classes by (4). Moreover, for all n the n-growing and n-shallow TRSs do not form super-classes by (1), which is not n-growing or n-shallow as the left-hand side is not linear.

The fourth class, defined by Jacquemard, Meyer, and Weidenbach [7], requires the rewrite rules to be of the following shapes.

$$f(g(x)) \to h(k(x))$$
$$f(x) \to t$$
$$t \to f(x)$$

with $f, g, h, k \in \Sigma_1$ and t a ground term. For all n the classes of n-growing and n-shallow TRSs do not form sub-classes by (3). Moreover, for all $n \neq 2$ the n-growing and n-shallow TRSs do not form super-classes by $f(g(x)) \rightarrow h(k(x))$. For $n = 2$ the n-growing do not form super-classes again by $f(g(x)) \rightarrow h(k(x))$, but the n-shallow TRSs do obviously form a super-class. Hence, Jacquemard, Meyer, and Weidenbach already proved that reachability, normalisation, and neededness are undecidable for 2-shallow TRSs.

7 Further Directions

At least two questions remain. First of all, are reachability, normalisation, and neededness undecidable for TRSs in which for each rewrite rule $l \rightarrow r$ we have for all $x \in \mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ that $d_l(x) = n$ and $d_r(x) < n$? Second, are reachability, normalisation, and neededness also undecidable for n-shallow and n-growing in case we require the TRSs to be orthogonal? It is highly likely that this is the case. However, our proofs are no longer applicable, as they make heavy use TRSs which are not orthogonal.

Acknowledgements. I would like to thank Jan Willem Klop, Aart Middeldorp, Femke van Raamsdonk, and Roel de Vrijer and the anonymous referees for their helpful comments and remarks.

References

1. F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1998.
2. H. Comon. Sequentiality, monadic second-order logic and tree automata. *Information and Computation*, 157(1–2):25–51, 2000.
3. I. Durand and A. Middeldorp. Decidable call by need computations in term rewriting. In W. McCune, editor, *Proc. of the 14th Int. Conf. on Automated Deduction (CADE-14)*, volume 1249 of *LNCS*, pages 4–18. Springer-Verlag, 1997.
4. G. Huet and J.-J. Lévy. Computations in orthogonal rewriting systems. In J.-L. Lassez and G. Plotkin, editors, *Computational Logic*, pages 395–443. MIT Press, 1991.
5. F. Jacquemard. Decidable approximations of term rewriting systems. In H. Ganzinger, editor, *Proc. of the 7th Int. Conf. on Rewriting Techniques and Applications (RTA '96)*, volume 1103 of *LNCS*, pages 362–376. Springer-Verlag, 1996.
6. F. Jacquemard. Reachability and confluence are undecidable for flat term rewriting systems. *Information Processing Letters*, 87(5):265–270, 2003.
7. F. Jacquemard, C. Meyer, and C. Weidenbach. Unification in extensions of shallow equational theories. Technical Report MPI-I-98-2-002, Max-Planck-Institut für Informatik, 1998.
8. E. L. Post. A variant of a recursively unsolvable problem. *Bulletin of the American Mathematical Society*, 52:264–268, 1946.
9. H. Seki, T. Takai, Y. Fujinaka, and Y. Kaji. Layered transducing term rewriting system and its recognizability preserving property. In S. Tison, editor, *Proc. of the 13th Int. Conf. on Rewriting Techniques and Applications (RTA 2002)*, volume 2378 of *LNCS*, pages 98–113. Springer-Verlag, 2002.