

Rewriting Transfinite Terms

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Dedicated to Roel de Vrijer on the occasion of his 60th birthday

Abstract

We define rewriting over terms with positions of transfinite length.

1 Introduction

During the time the first author of this paper was a Ph.D. student at the Vrije Universiteit, one of Roel’s international Master students was writing his thesis on meaningless terms in (infinitary) λ -calculus [2]. At one point Roel suggested that the student should most certainly consider infinite terms of the form $xx \cdots$, consisting of an infinite number of variables and associating to the left as usual (see Figure 1). Unfortunately, as pointed out soon afterwards by the first author such a term is not a valid infinite λ -term, as the left-most variables will occur at infinite depth.

Inspired by Roel’s suggestion, we will introduce terms—the so-called transfinite terms—which allow for variables to occur at infinite depth. To support the definition, we will also define positions that are of arbitrary ordinal length, and not just of finite length. After having defined the transfinite terms we will attempt to define an infinitary rewrite relation over them.

Besides being inspired by Roel, there are two other reasons for considering terms which allow for variables, and also function symbols, to occur at infinite depth. Both of these reasons find their origin in infinitary rewriting and some of its shortcomings.

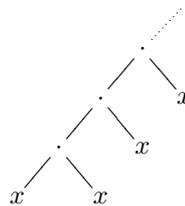


Figure 1: The infinite term $xx \cdots$.

The first reason is the lack of confluence in orthogonal infinitary rewriting [9]. The well-known counterexample considers the following orthogonal set of rules:

$$\left\{ \begin{array}{l} c \rightarrow f(g(c)) \\ f(x) \rightarrow x \\ g(x) \rightarrow x \end{array} \right.$$

Given these rules, the following two co-initial infinite reductions starting in c are then considered:

$$f^\omega \dots \leftarrow f(c) \leftarrow f(g(c)) \leftarrow c \rightarrow f(g(c)) \rightarrow g(c) \rightarrow \dots g^\omega.$$

The reduction to the left alternates between contracting the only occurring $c \rightarrow f(g(c))$ -redex and the only occurring $g(x) \rightarrow x$ -redex. The reduction to the right similarly contracts $c \rightarrow f(g(c))$ -redexes and $f(x) \rightarrow x$ -redexes. The terms f^ω and g^ω can never be reduced to each other; they reduce only to themselves (in both finitely and infinitely many steps). To see why this is the case, note that in both depicted reductions the $c \rightarrow f(g(c))$ -redex is “pushed down” to depth ω ; the redex is then lost, because such positions are not considered in infinitary rewriting. In the rewriting format defined in this paper, the $c \rightarrow f(g(c))$ -redex will precisely occur at depth ω after ω steps and this will actually restore confluence in of case of the rules give above. It is our hope that this result generalises to all orthogonal systems. However, we do not attempt prove this.

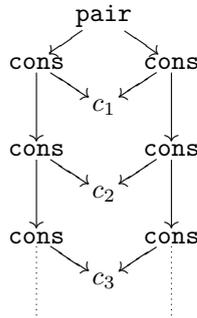


Figure 2: An infinite DAG.

The second reason for studying terms with positions of arbitrary ordinal length is related the representation of directed acyclic graphs (DAGs) as terms with let-binders, i.e. as terms with subexpressions of the form **let** $x = s$ **in** t . It is folklore that each *finite* DAG can be represented by *finite* terms with let-binders and vice versa, see e.g. [3]. Unfortunately, a similar relation does not exist between *infinite* DAGs and *infinite* terms. Consider e.g. the DAG depicted in Figure 2, where **pair** is a pairing operator and where **cons** is the usual list constructor. Intuitively, and using the usual short-hand notation

for lists, this DAG should correspond to the following infinite term:

$$\text{let } x_1 = c_1 \text{ in let } x_2 = c_2 \text{ in } \dots \text{pair}([x_1, x_2, \dots], [x_1, x_2, \dots]).$$

However, given the infinite number of let-binders—one to represent each c_i -subgraph of the DAG—it follows that **pair** $([x_1, x_2, \dots], [x_1, x_2, \dots])$ must occur at depth ω , and we have moved again beyond the scope of infinitary rewriting.

Outline. In the remainder of the paper we proceed as follows: In Section 2 we define terms in which function symbols and variables may occur at arbitrary ordinal depth. To this end we also define positions of transfinite length. In Section 3 we define rewrite rules and rewrite steps over the terms defined in the previous section. Finally, in Section 4 we attempt to define infinite reductions and in Section 5 we conclude.

Notation. Throughout, we assume that Σ is a set of function symbols each of which has *finite* arity and that X is a sufficiently large set of variables. We denote the first infinite ordinal by ω and arbitrary ordinals by α, β, γ , and so on. Moreover, we denote the first non-denumerable cardinal by \aleph_1 . The restriction of a function f with domain S to domain $T \subseteq S$ is denoted by $f \upharpoonright_T$.

2 Transfinite Terms

In infinitary rewriting—and also in finitary rewriting—a term can be defined as a map from a set of positions to the elements of $\Sigma \cup X$ [6, 4, 10]. Such a map must satisfy a number of natural constraints regarding the assumed signature and the positions in the domain of the map. Here, we extend this definition to include terms with transfinite positions. Such positions are maps from *arbitrary* ordinals to \mathbb{N} , this in contrast to the usual positions which are maps from *finite* ordinals to the natural numbers, i.e. which are finite strings over \mathbb{N} .

Definition 2.1 (Transfinite Positions). A *transfinite position* p is a map $p : \alpha \rightarrow \mathbb{N}$ with α an ordinal. The *length* of p , denoted $|p|$, is α .

We denote the unique transfinite position $\emptyset \rightarrow \mathbb{N}$ —the root position—by ϵ . The transfinite positions fully specified by $1 \mapsto 1, 1 \mapsto 2, 1 \mapsto 3, \&c.$ are denoted by $1, 2, 3, \&c.$ Arbitrary transfinite positions are denoted by p, q , and so on.

Given the above definition, the prefix relation over transfinite positions p and q , denoted $p < q$, can be defined as:

$$p < q \iff |p| < |q| \text{ and } q \upharpoonright_{|p|} = p.$$

Recall that the *lengths* of p and q are in fact the domains of the *maps* p and q . Hence, as for any two ordinals α and β we have $\alpha < \beta$ iff $\alpha \subseteq \beta$, it follows that the restriction used to defined the prefix relation is proper. If neither $p \leq q$ nor $q \leq p$, then p and q are *parallel*, which we write as $p \parallel q$.

The concatenation of transfinite positions p and q , denoted $p \cdot q$, is the operation with ϵ as unit element and which for each $|p| = \alpha$ and $|q| = \beta$ yields the following map from $\alpha + \beta$ to \mathbb{N} :

$$p \cdot q(a) = \begin{cases} p(a) & \text{if } a \leq \alpha \\ q(b) & \text{if } a = \alpha + b \end{cases}$$

When restricted to finite ordinals, transfinite positions are isomorphic to finite strings over the natural numbers. Using this fact, it is easy to show that the prefix ordering and concatenation as defined above correspond to the usual (finitary) definitions in case the domain is restricted as such.

Example 2.2. We have that $\epsilon, 1, 1 \cdot 1$, and $1 \cdot 2$ are transfinite positions with $\epsilon < 1$ and $1 \cdot 1 \parallel 1 \cdot 2$. Let $1^\alpha : \alpha \rightarrow \mathbb{N}$ be such that $1^\alpha(a) = 1$, where 1^0 is assumed to be ϵ . For each ordinal α the map 1^α is a transfinite position. In addition, $1^\alpha < 1^\beta$ for each $\alpha < \beta$ and $1^\gamma \cdot 1^\delta = 1^{\gamma+\delta}$ for arbitrary γ and δ .

Before we continue to define transfinite terms, we give one more definition related to positions, which will come of use when defining reductions in Section 4.

Definition 2.3 (Limits of Sequences of Transfinite Positions). Let $(p_\beta)_{\beta < \alpha}$ be a sequence of transfinite positions, with α a limit ordinal. Then, $\lim_{\beta \rightarrow \alpha} p_\beta = p$ for some transfinite position p iff

1. there exists a $\kappa < \alpha$ such that $p_\delta = p$ for all $\kappa < \delta < \alpha$, or
2. for all $q < p$ there exists a $\kappa < \alpha$ such that $q < p_\delta < p$ for all $\kappa < \delta < \alpha$.

Given the fact that weakly increasing sequences of ordinals have unique limits in case the sequence has an upper bound, it follows easily that the limit of a sequence of transfinite positions is unique in case it exists.

We now define transfinite terms as maps from transfinite positions to the elements of $\Sigma \cup X$. For the duration of the definition we assume variables to be nullary function symbols, i.e. constants.

Definition 2.4 (Transfinite Terms). A *transfinite term* is a map $t : P \rightarrow \Sigma \cup X$ with P a non-empty set of transfinite positions such that

- if $p \in P$, then $q \in P$ for all $q < p$,
- if $t(p)$ is a function symbol of arity n , then $p \cdot i \in P$ iff $1 \leq i \leq n$, and
- if p has limit ordinal length and $q \in P$ for all $q < p$, then $p \in P$.

The first and second clause ensure, respectively, that transfinite terms do not have ‘disconnected’ subterms and that the arity of each function symbol is factored in. The third clause ensures that we do not have transfinite terms where, eloquently put, “the bottom falls out” in certain places. In other words, for any path we can follow from the root of a transfinite term downwards, there will always be a properly defined position that either coincides with the final position of the path or that goes beyond any position in the path.

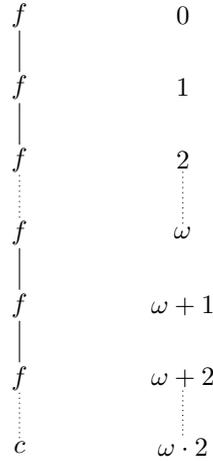


Figure 3: The transfinite term $f^{\omega \cdot 2}(c)$ and the depth at which each of its function symbol occurs.

Remark that the definition of (finite) terms limits the above definition by requiring P to be finite; this voids the third clause of the definition. The definition of infinite terms limits the above definition by dropping the last clause and requiring that each position is a map from a *finite* ordinal to $\Sigma \cup X$.

Example 2.5. Suppose we have the set $P = \{1^\alpha \mid \alpha \leq \omega\}$ with 1^α as in Example 2.2. Moreover, suppose our signature contains a function symbol f of arity 1 and a constant c . Mapping 1^n to f for all $n \in \mathbb{N}$ and mapping 1^ω to c defines a transfinite term. Informally we write this term as $f^\omega(c)$. In a similar way, we can define the transfinite term $f^{\omega \cdot 2}(c)$ (see Figure 3), or even $f^\alpha(c)$, where α is an arbitrary ordinal.

Let $t : P \rightarrow \Sigma \cup X$ be a transfinite term, we define the set of transfinite positions of t to be the domain of t and denote it by $\mathcal{P}os(t)$. The set of transfinite positions at which variables, respectively function symbols, occur is denoted by $\mathcal{P}os_X(t)$, respectively $\mathcal{P}os_\Sigma(t)$. Moreover, we write $root(t) = t(\epsilon)$, i.e. by $root(t)$ we denote the function symbol at the root of t . The root symbol always exists, as P is non-empty and closed under the prefix relation over transfinite positions.

The transfinite subterm of t at position $p \in \mathcal{Pos}(t)$, denoted $t|_p$, is the map with domain $\{q \mid p \cdot q \in \mathcal{Pos}(t)\}$ such that $t|_p(q) = t(p \cdot q)$. The replacement of the transfinite subterm at position $p \in \mathcal{Pos}(t)$ by the transfinite term s , denoted $t[s]_p$, is the map with domain

$$\{q \mid q \in \mathcal{Pos}(t) \text{ and } p \not\preceq q\} \cup \{p \cdot q \mid q \in \mathcal{Pos}(s)\}$$

such that:

$$t[s]_p(q) = \begin{cases} t(q) & \text{if } p \not\preceq q \\ s(r) & \text{if } q = p \cdot r \end{cases}$$

Given a substitution σ , i.e. a map from a set of variables to a set of transfinite terms, $\sigma(t)$ is defined as the map with domain

$$\begin{aligned} & \{p \mid p \in \mathcal{Pos}(t) \text{ and } t(p) \notin \mathcal{Dom}(\sigma)\} \\ & \cup \{p \cdot q \mid t(p) = x \text{ with } x \in \mathcal{Dom}(\sigma) \text{ and } q \in \mathcal{Pos}(\sigma(x))\} \end{aligned}$$

such that:

$$\sigma(t)(p) = \begin{cases} t(p) & \text{if } t(p) \notin \mathcal{Dom}(\sigma) \\ \sigma(x)(r) & \text{if } p = q \cdot r \text{ and } t(q) = x \text{ with } x \in \mathcal{Dom}(\sigma) \end{cases}$$

We call σ a *renaming* in case $\sigma(x)$ is a variable for each x .

It is easy to see that taking subterms, replacing subterms, and carrying out substitutions all yield well-defined transfinite terms. Moreover, the definitions correspond to the usual ones when we restrict ourselves to finite terms.

To wrap this section up, note that contexts can be defined as usual, i.e. by extending the signature with a fresh constant \square . We denote one-hole contexts by $C[\square]$, $D[\square]$, and so on. Replacing the hole in $C[\square]$ by a transfinite term t , which is denoted by $C[t]$, is defined as $C[t] = (C[\square])[t]_p$, assuming that $(C[\square])|_p = \square$.

3 Transfinite Term Rewriting Systems

With the help of the definitions from the previous section it now becomes a trivial matter to extend the usual definitions of rewrite rules and rewrite steps to transfinite terms. To keep things straightforward, we will assume that both left-hand and right-hand sides of rewrite rules are finite.

Definition 3.1 (Transfinite Rewrite Rules and Term Rewriting Systems). A *transfinite rewrite rule* is a pair of transfinite terms (l, r) each with a finite set of positions and denoted $l \rightarrow r$ such that l is not a variable and such that each variable which occurs in r also occurs in l .

A *transfinite Term Rewriting System (tTRS)* is a pair (Σ, R) with Σ a signature each element of which has finite arity and with R a set of transfinite rewrite rules.

As usual, we call l the left-hand side of the rule $l \rightarrow r$ and r the right-hand side.

Example 3.2. Suppose $f(x, y)$, $g(y)$, and x are transfinite terms. Then, the following are transfinite rewrite rules:

$$\begin{aligned} f(x, y) & \rightarrow g(x) \\ f(x, y) & \rightarrow x \end{aligned}$$

The pair $x \rightarrow x$ is not a transfinite rewrite rule, as the left-hand side is a variable. The pair $g(y) \rightarrow f(x, y)$ is not a transfinite rewrite rule either, as x occurs right but not left.

We can now define transfinite rewrite steps.

Definition 3.3 (Transfinite Rewrite Steps). A *transfinite rewrite step* is a pair of transfinite terms (s, t) denoted $s \rightarrow t$ and adorned with a one-hole context $C[\square]$, a transfinite rewrite rule $l \rightarrow r$, and a substitution σ such that $s = C[\sigma(l)]$ and $t = C[\sigma(r)]$. The transfinite term $\sigma(l)$ is called an $l \rightarrow r$ -redex and it *occurs* at position p and *depth* $|p|$ in s , where p is the position of the hole in $C[\square]$.

Example 3.4. Suppose we have the following three transfinite rewrite rules:

$$\left\{ \begin{array}{l} c \rightarrow d \\ f(x) \rightarrow x \\ g(x, x) \rightarrow x \end{array} \right.$$

Given the transfinite term $f^\omega(c)$, we have $f^\omega(c) \rightarrow f^\omega(c)$, contracting any of the $f(x) \rightarrow x$ -redexes in $f^\omega(c)$. Note that we do *not* have $f^\omega(c) \rightarrow f^{\omega+1}(c)$ and that, in fact, for each $k > 0$ and $l \in \mathbb{N}$ and rule $f^k(x) \rightarrow f^l(x)$ we have $f^\omega(c) \rightarrow f^\omega(c)$ when we contract an arbitrary $f^k(x) \rightarrow f^l(x)$ -redex in $f^\omega(c)$.

Returning to our three rules above, we also have $f^\omega(c) \rightarrow f^\omega(d)$, contracting the $c \rightarrow d$ -redex at depth ω . Moreover, $g(f^\alpha(c), f^\alpha(c)) \rightarrow f^\alpha(c)$ for every ordinal α , contracting in $g(f^\alpha(c), f^\alpha(c))$ the $g(x, x) \rightarrow x$ -redex at depth 0.

Finally, for the transfinite rule $f(x) \rightarrow g(x)$ we have $f^\omega(c) \rightarrow f^\omega(g(f^\omega(c)))$ when we contract the $f(x) \rightarrow g(x)$ -redex at depth n in $f^\omega(c)$. Note that we do *not* have $f^\omega(c) \rightarrow f^\omega(g(c))$.

The following is trivial:

Proposition 3.5 (Closure under Contexts and Substitutions). *If $s \rightarrow t$ is a transfinite rewrite step, then so is $C[\sigma(s)] \rightarrow C[\sigma(t)]$ for arbitrary (transfinite) contexts $C[\square]$ and (transfinite) substitutions σ .*

To end this section we define descendants across rewrite steps, we employ this definition later on when we define reductions of transfinite length.

Definition 3.6 (Descendants). Let $u : C[\sigma(l)] \rightarrow C[\sigma(r)]$ be a transfinite rewrite step whose redex occurs at position p . If $q \in \mathcal{Pos}(C[\sigma(l)])$, then the set of *descendants* of q across u , denoted q/u , is defined as follows:

$$q/u = \begin{cases} \{p \cdot p_r \cdot q' \mid \text{root}(r|_{p_r}) = x\} & \text{if } q = p \cdot p_l \cdot q' \text{ and } \text{root}(l|_{p_l}) = x \in X \\ \{p\} & \text{if } q = p \cdot p_l \text{ with } \text{root}(l|_{p_l}) \notin X \\ \{q\} & \text{if } p \not\preceq q \end{cases}$$

With the help of the definitions presented at the end of the previous section it is not difficult to show that descendants across rewrite steps are well-defined. Moreover, note that any position that occurs in the redex pattern of a contracted redex descends to the root position of that redex when the redex is contracted.

4 Transfinite Reductions

Given the above definitions, finite reductions over transfinite terms are easily defined. However, as in infinitary rewriting [5, 9, 8], defining transfinite reductions is rather more complicated.

Since our definition of terms allows for positions of arbitrary ordinal length, it seems natural to require from our definition of transfinite reductions that it offers the ability to “push down” subterms from some finite depth to limit ordinal depth and the ability to “pull up” subterms from limit ordinal depth to some lesser depth. Let us illustrate these two concepts.

Suppose we have the transfinite term c and the transfinite rewrite rule $c \rightarrow f(c)$. Continuously contracting $c \rightarrow f(c)$ -redexes yields the following:

$$c \rightarrow f(c) \rightarrow f^2(c) \rightarrow \dots \rightarrow f^n(c) \rightarrow f^{n+1}(c) \rightarrow \dots . \quad (1)$$

Since the subterm c occurs at an increasingly greater depths and since the function symbol that occurs at every other depth becomes f after a finite number of steps, it seems natural to require the final term of the above reduction to be the transfinite term $f^\omega(c)$ from Example 2.5. That is, c is “pushed down” from finite depth 0 to infinite depth ω in ω steps. After these ω steps the same procedure can be repeated with regard to the subterm c at position 1^ω in $f^\omega(c)$ which by identical reasoning should yield the transfinite term $f^{\omega \cdot 2}(c)$.¹

Now consider the transfinite term $g^\omega(c)$ and the transfinite rewrite rule $g(x) \rightarrow x$. Continuously contracting root redexes yields the following:

$$g^\omega(c) \rightarrow g^\omega(c) \rightarrow \dots \rightarrow g^\omega(c) \rightarrow g^\omega(c) \rightarrow \dots . \quad (2)$$

Intuitively, every $g(x) \rightarrow x$ -redex in $g^\omega(c)$ will eventually be contracted along this reduction. Hence, it makes sense to define the final transfinite term of this reduction to be the only part of $g^\omega(c)$ which is not touched at all along the reduction, i.e. to define it as c . Hence, we “pull up” c from depth ω to depth 0.

We will now attempt to find a proper definition of transfinite reductions over transfinite terms that matches the above intuition.

4.1 A Topological Defeat

In ‘ordinary’ infinitary rewriting reductions are essentially continuous maps from an ordinal equipped with the order topology to the set of infinite terms equipped with the topology obtained through metric completion of the usual distance metric on finite terms [5, 9]. Hence, a natural question to ask is, if this definition can be re-used in the current context to obtain a useful definition reductions that satisfies both the “push down” and “pull up” property. Jumping the gun: This turns out not to be the case.

Let us consider why the answer to the above question is negative. To this end we define a topology over the transfinite terms.² We do this by means of defining a base [7], where a *base* is a family \mathcal{B} of subsets of a topological space, such that for every open set O there is a subset $\mathcal{B}' \subseteq \mathcal{B}$ for which $\bigcup_{B \in \mathcal{B}'} B = O$.

¹The complete reduction of length $\omega \cdot 2$ also motivates why we do not limit the ordinals in Definition 2.1 to be, say, at most ω .

²Technically, there is no set containing all transfinite terms, as the definition is based on positions of arbitrary ordinal length and as there is no set containing all ordinals. In practice, however, we are only interested in classes of terms (reductions) that properly form sets.

Definition 4.1. Let t be a transfinite term and let α either 0 or a *successor* ordinal such there is a $p \in \mathcal{Pos}(t)$ with $|p| = \alpha$. Define the set $B_{t,\alpha}$ to be:

$$\{s \mid \forall q \in \mathcal{Pos}(t) \text{ with } |q| \leq \alpha : q \in \mathcal{Pos}(s) \text{ and } s|_q = t|_q\}$$

and the family \mathcal{B} to be:

$$\{B_{t,\alpha} \mid \text{for all } t \text{ and appropriate } \alpha\} \cup \emptyset$$

Each set $B_{t,\alpha}$ contains exactly those transfinite terms that are equal to t up to depth α . Note that $B_{t,\alpha} = \{t\}$ in case we have for all $p \in \mathcal{Pos}(t)$ that $|p| \leq \alpha$ with α either 0 or a successor ordinal. Limit ordinals are excluded from the definition to ensure for each transfinite term such as $f^\omega(c)$ that there always is some finite term close to it.

For any $B_{t,\alpha}$ and $B_{t',\alpha'}$ we have either $B_{t,\alpha} \cap B_{t',\alpha'} = \emptyset$ or $B_{t,\alpha} \cap B_{t',\alpha'} = B_{s,\beta}$ with $s \in B_{t,\alpha} \cap B_{t',\alpha'}$ and $\beta = \max\{\alpha, \alpha'\}$. Hence, there exists a topology over the transfinite terms with \mathcal{B} as base.

Remark 4.2. The above base finds its origins in the usual topology over the set of infinite terms. Replacing the ordinals in the above definition by natural numbers, we obtain a base for the infinite terms. It is not difficult to see that the topology defined as such corresponds to the topology generated by the following metric:

$$d(s, t) = \begin{cases} 0 & \text{if } s = t \\ s^{-k} & \text{if } k = \min\{|p| \mid p \in \mathcal{Pos}(s) \cap \mathcal{Pos}(t) \text{ and } s|_p \neq t|_p\} \end{cases}$$

which is the usual metric over infinite terms [1].³

Recall that the *order topology* over any ordinal α has as its base the set of all open intervals. That is, the base consists of all of the following sets:

$$\begin{aligned} (\beta, \gamma) &= \{\kappa \in \alpha \mid \beta < \kappa < \gamma\} \\ (\beta, \rightarrow) &= \{\kappa \in \alpha \mid \kappa > \beta\} \\ (\leftarrow, \beta) &= \{\kappa \in \alpha \mid \kappa < \beta\} \\ (\leftarrow, \rightarrow) &= \alpha \end{aligned}$$

A reduction can now potentially be defined as a continuous map from an ordinal equipped with the order topology to the transfinite terms equipped with the topology defined by \mathcal{B} above. Unfortunately, there are two issues that make this definition unusable. First, for any reduction of limit ordinal length defined as such, it turns out that the final term of the reduction is *not* uniquely determined by the previous terms or cannot be the term we want it to be. Second, the “pull up” property is inherently non-continuous and, hence, not covered by the definition. We will now illustrate both phenomena.

Consider the reduction from (1) and recall that a map ϕ is continuous iff for every open set O we have that $\phi^{-1}[O]$ too is open [7]. As is easy to check, the map $\phi(\alpha) = f^\alpha(c)$ with domain $\omega + 1$ is continuous and represents the above reduction extended with the expected transfinite term $f^\omega(c)$. Unfortunately, this

³This metric does not carry over to transfinite terms, as 2^{-k} is only defined in case k is a *finite* ordinal, i.e. a natural number.

is not the only possible way to extend the reduction when the only requirement is that the resulting map should be continuous: It is easily checked that any map ψ will do as long as $\psi \upharpoonright_\omega = \phi \upharpoonright_\omega$ and $\psi(\omega) = f^\omega(t)$ with t either a constant or a variable. Hence, the extension of the reduction with $f^\omega(c)$ is not unique.

Now consider the transfinite rewrite rule $f(x) \rightarrow g(f(x))$ and the transfinite term $f(h^\omega(c))$. We have the following reduction:

$$\begin{aligned} f(h^\omega(c)) &\rightarrow g(f(h^\omega(c))) \rightarrow g^2(f(h^\omega(c))) \rightarrow \dots \\ &\rightarrow g^n(f(h^\omega(c))) \rightarrow g^{n+1}(f(h^\omega(c))) \rightarrow \dots \end{aligned} \quad (3)$$

Taking into account the “push down” property, one would expect the final transfinite term of this reduction to be $g^\omega(f(h^\omega(c)))$. Unfortunately, the only continuous maps extending this reduction with one additional term are those maps ϕ with $\phi(\alpha) = g^\alpha(f(h^\omega(c)))$ for all $\alpha < \omega$ and $\phi(\omega) = g^\omega(t)$ such that t is either a constant or a variable. Obviously, the term $g^\omega(f(h^\omega(c)))$ is not such a transfinite term.

Although the above issues with the topology and the “push down” property might potentially be alleviated by adapting the employed topology, this is impossible in the case of the “pull up” property. This property interacts with continuity in a more subtle way. To see this, consider again the reduction from (2). In case of this reduction we would like the following map with domain $\omega + 1$ to be only possible continuous map that represents the reduction and extends it with one additional transfinite term:

$$\phi(\alpha) = \begin{cases} g^\omega(c) & \text{if } \alpha < \omega \\ c & \text{if } \alpha = \omega \end{cases}$$

Within the current topology, this map is not continuous. In fact, and fatal to any topological approach based solely on continuous maps, it is clear that in which ever way we define our topology the map $\psi(\alpha) = g^\omega(c)$ with domain $\omega + 1$ will *always* be continuous too, as for each neighbourhood of $g^\omega(c)$ we have that $g^\omega(c)$ is in that neighbourhood. Hence, we need to move away from a topological approach employing continuous maps.

4.2 A Pyrrhic Victory

Topologies out of the picture, what about adapting an alternative definition of reductions from infinitary rewriting? Such an alternative definition is easily given both in the case of weakly convergent infinitary rewriting, i.e. the kind that does not take into account the depths of the contracted redexes, and in the case of strongly convergent infinitary rewriting, i.e. the kind where at each depth only finitely many redexes may be contracted.

Definition 4.3 (Infinitary Rewriting). A *transfinite reduction* over infinite terms is a sequence of infinite terms $(t_\beta)_{\beta < \alpha+1}$ such that $t_\beta \rightarrow t_{\beta+1}$ for all $\beta < \alpha$. The reduction is *weakly convergent* if for every limit ordinal $\gamma < \alpha + 1$ and for all $p \in \mathcal{Pos}(t_\gamma)$ there exists a $\kappa < \gamma$ such that $\text{root}(t_\gamma|_p) = \text{root}(t_\beta|_p)$ for all $\kappa \leq \beta < \gamma$. The reduction is *strongly convergent* if it is weakly convergent and if for every limit ordinal $\gamma < \alpha + 1$ the depth of the contracted redexes in $(t_\beta)_{\beta < \gamma}$ tends to infinity.

The constraint on weakly convergent reductions implies that these reductions are in fact continuous maps in the context of infinitary rewriting. The additional constraint on strongly convergent reductions implies that only a finite number of redexes can be contracted at each depth. Remark that the additional constraint on strongly convergent reductions can alternatively be formulated as follows: For every position $p \in \mathcal{Pos}(t_\gamma)$, eventually all redexes contracted in $(t_\beta)_{\beta < \gamma}$ either occur parallel to p or strictly below p , i.e. only at (finite) positions q with $q \not\leq p$.

Below we will attempt to extend the definition of strongly convergent reductions to rewriting with transfinite terms. The reason we do not consider weakly convergent reductions is that we would also like to obtain a proper notion of descendants and in infinitary rewriting such a notion only exists in the strongly convergent setting [11]. We have to be careful, though. Consider e.g. the transfinite rewrite rule $f(x) \rightarrow g(f(h(x)))$ and the reduction

$$\begin{aligned} f(c) \rightarrow g(f(h(c))) \rightarrow g^2(f(h^2(c))) \rightarrow \dots \\ \rightarrow g^n(f(h^n(c))) \rightarrow g^{n+1}(f(h^{n+1}(c))) \rightarrow \dots \end{aligned}$$

Using the “push down” property and given the fact that the subterm starting with f converges to $f(h^\omega)$, it makes sense to define the unique transfinite term extending this reduction to be $g^\omega(f(h^\omega(c)))$. However, although each h in $g^\omega(f(h^\omega(c)))$ must descent from a function symbol h created in some step along the above reduction, it is—by the infinite nature of the reduction—impossible to tell in which step which h was created. Hence, to obtain a proper notion of descendants for rewriting with transfinite terms we want to rule out reductions such as the above.

Remark 4.4. The above reduction is related to certain reductions without a properly defined limit: Consider the transfinite rewrite rules $f(x) \rightarrow g(f(h_1(x)))$ and $f(x) \rightarrow g(f(h_2(x)))$ (note the subscripts of h). We have the following:

$$\begin{aligned} f(c) \rightarrow g(f(h_1(c))) \rightarrow g^2(f(h_2(h_1(c)))) \rightarrow \dots \\ \rightarrow g^{2 \cdot n}(f((h_2 h_1)^n(c))) \rightarrow g^{2 \cdot n+1}(f(h_1((h_2 h_1)^n(c)))) \\ \rightarrow g^{2 \cdot (n+1)}(f((h_2 h_1)^{n+1}(c))) \rightarrow \dots \end{aligned}$$

Although it is obvious that any limit of this reduction must have the form $g^\omega(f(t))$ where t consists of an infinite alternation of h_1 and h_2 followed by c , we do not know if this alternation starts with either h_1 or with h_2 . Hence, there is no limit which constitutes a properly defined transfinite term.

We will now attempt to extend the definition of strongly convergent reductions to transfinite terms, taking into account the “push down” and “pull up” properties. To start, we define transfinite reductions:

Definition 4.5 (Transfinite Reductions over Transfinite Terms). A *transfinite reduction* is a sequence of transfinite terms $(t_\beta)_{\beta < \alpha}$ such that $t_\beta \rightarrow t_{\beta+1}$ for all $\beta < \alpha$. If $\alpha = \alpha' + 1$, then the sequence is *closed* and of length α' . If α is a limit ordinal, then the sequence is *open* and of length α .

Obviously, the above definition alone does not suffice since no relation is established between the transfinite terms with limit ordinal indices and the

terms preceding them. As in infinitary rewriting, we want the terms with limit ordinal indices to be uniquely determined by the terms preceding them.⁴

Before we can continue to define strongly convergent reductions, which do relate limit terms with those terms preceding them, we need the ability to track those redex contractions responsible for “pulling up” and “pushing down” subterms. To this end we define the following:

Definition 4.6 (Outermost Pull-up/Push-down Property). Let $(t_\beta)_{\beta < \alpha}$ be a transfinite reduction and let $(p_\beta)_{\kappa \leq \beta < \alpha}$ and $(q_\beta)_{\kappa \leq \beta < \alpha}$ be sequences of transfinite positions such that $p_\beta, q_\beta \in \mathcal{Pos}(t_\beta)$ for all $\kappa \leq \beta < \alpha$. The pair of sequences satisfies the *outermost pull-up/push-down property* (*outermost pp-property*) for a term s if for each limit ordinal $\kappa < \gamma < \alpha$ it holds that $\lim_{\beta \rightarrow \gamma} p_\beta = p_\gamma$ and if for all $\kappa \leq \beta < \alpha$ it holds that $t_\beta|_{p_\beta} = \sigma(s)$ for some substitution σ and:

- the $l \rightarrow r$ -redex contracted in $t_\beta \rightarrow t_{\beta+1}$ either occurs parallel to p_β such that $p_{\beta+1} = p_\beta$ and $q_{\beta+1} = q_\beta$ or it occurs at a position $p_\beta \cdot p'_\beta$ with $q_\beta \notin \{p_\beta \cdot p'_\beta \cdot q \mid q \in \mathcal{Pos}_\Sigma(l) - \{\epsilon\}\}$ such that:
 - if $p'_\beta = \epsilon$, then $p_{\beta+1} = p_\beta \cdot p''_\beta$ and $r|_{p''_\beta} = \tau(s)$ with τ a renaming,
 - if $p'_\beta \neq \epsilon$, then $p_{\beta+1} = p_\beta$ and $p'_\beta \notin \mathcal{Pos}_\Sigma(s)$,

and if $q_\beta = p_\beta \cdot q'_\beta \cdot q''_\beta$ with $q'_\beta \in \mathcal{Pos}_X(s)$, then $q_{\beta+1} = p_{\beta+1} \cdot q'_\beta \cdot q''_\beta$, and

- there is a $\beta < \delta < \alpha$ such that $t_\delta \rightarrow t_{\delta+1}$ contracts a redex at the transfinite position p_δ .

The redexes contracted at the transfinite positions in $(p_\beta)_{\kappa \leq \beta < \alpha}$ will be those responsible for “pushing down” and “pulling up” the subterm that occurs at position q_κ in t_κ , where the requirements on q_β ensure that we can properly talk about such a subterm. We call the property outermost, because we require—in the first clause—that each contracted redex $t_\beta \rightarrow t_{\beta+1}$ either occurs at a position parallel to p_β or a position with p_β as prefix.

The function symbols from s are continuously ‘re-created’ along the reduction. We single out these function symbols, because we want them to re-occur in the limit of the reduction. This behaviour can already be observed in our examples: In the reduction from (1), respectively (3), we have that s is c , respectively $f(x)$. In the reduction from (2) no function symbols are continuously re-created and s will be a variable. Note that, since eventually a redex is contracted at a position in $(p_\beta)_{\kappa \leq \beta < \alpha}$, we have that s is uniquely determined by $(p_\beta)_{\kappa \leq \beta < \alpha}$.

The requirement that $p'_\beta \notin \mathcal{Pos}_\Sigma(s)$ in case $p'_\beta \neq \epsilon$ ensures that the function symbols from s cannot be erased before being re-created. Intuitively, it is unclear if a function symbol from s should re-occur in the limit of a reduction in case it is each time being erased before being re-created. Hence, we disallow this.

Finally, the requirement that $q_{\beta+1} = p_{\beta+1} \cdot q'_\beta \cdot q''_\beta$ in case $q_\beta = p_\beta \cdot q'_\beta \cdot q''_\beta$ with $q'_\beta \in \mathcal{Pos}_X(s)$, ensures that there is a uniquely determined transfinite position in the limit of the reduction at which the subterm at q_κ will occur. Note that this requirement avoids problems related to those that cause difficulties with descendants in weakly convergent infinitary rewriting [9, 11]. With respect to

⁴Although never stated explicitly in infinitary rewriting, the validity of this property follows immediately by requiring reductions to be continuous maps in a complete metric space.

weakly convergent infinitary rewriting, consider the rule $f(x, y) \rightarrow f(y, x)$ and recall the following reduction of length ω :

$$f(c, c) \rightarrow f(c, c) \rightarrow \cdots \rightarrow f(c, c) \rightarrow \cdots f(c, c).$$

It is unclear which c in the final term of the reduction descends from which c in the initial term.

Example 4.7. Consider the reductions from (1), (2), and (3). In case of the reduction from (1), respectively (2), we have that the pair of sequences each of which is $(1^\beta)_{0 \leq \beta < \omega}$, respectively $(\epsilon)_{0 \leq \beta < \omega}$, has the outermost pp-property. In case of the reduction from (3), the sequences $(1^\beta)_{0 \leq \beta < \omega}$ and $(1^\beta \cdot 1)_{0 \leq \beta < \omega}$ form a pair with the outermost pp-property.

We are now in a position to define strongly convergent (and strongly continuous) reductions over transfinite terms. With regard to the definition, recall that any transfinite position that occurs in the redex pattern of a contracted redex descends to the root position of that redex when the redex is contracted.

Definition 4.8 (Strongly Convergent Reductions over Transfinite Terms). Let $(t_\beta)_{\beta < \alpha}$ be a transfinite reduction over transfinite terms. The reduction is *strongly continuous* if for every limit ordinal $\gamma < \alpha + 1$ and every $p \in \mathcal{Pos}(t_\gamma)$ there exists a $\kappa < \gamma$ such that exactly one of the following holds:

- all redexes contracted in $(t_\beta)_{\kappa \leq \beta < \gamma}$ occur either parallel or strictly below p and $\text{root}(t_\beta|_p) = \text{root}(t_\gamma|_p)$, or
- there exists a pair of sequences $((p_\beta)_{\kappa \leq \beta < \gamma}, (q_\beta)_{\kappa \leq \beta < \gamma})$ with the outermost pp-property for a term s such that for every $\kappa \leq \beta < \gamma$:
 - q_δ , is a descendant of q_β for all $\beta < \delta < \gamma$, and
 - for every p'_β with either $p'_\beta \geq p_\beta$ if $q_\beta = p_\beta$ or $q_\beta > p'_\beta \geq p_\beta$ if $q_\beta > p_\beta$ there exists a $\beta < \delta < \gamma$ such that p'_δ is the only descendant of p'_β ,

and $p = p' \cdot p''$ with $\lim_{\beta \rightarrow \gamma} p_\beta = p'$ such that either:

- $p'' \in \mathcal{Pos}_\Sigma(s)$ and $\text{root}(t_\gamma|_p) = \text{root}(s|_{p''})$, or
- $p'' = q' \cdot q''$ with $q' = q'_\kappa$ such that $q'_\kappa \in \mathcal{Pos}_X(s)$ and $q_\kappa = p_\kappa \cdot q'_\kappa \cdot q''_\kappa$, and $\text{root}(t_\gamma|_p) = \text{root}(s_{\gamma'}|_{q''})$ where $s_{\gamma'}$ is the final term of the reduction obtained from $(t_\beta|_{q_\beta})_{\beta < \gamma+1}$ by removing all empty steps.

The reduction is *strongly convergent* if it is strongly continuous and closed.

We denote a *strongly convergent* reduction of ordinal length α , respectively of ordinal length at most α , by $s \twoheadrightarrow^\alpha t$, respectively $s \twoheadrightarrow^{\leq \alpha} t$. A *strongly convergent* reduction of arbitrary ordinal length is denoted by $s \twoheadrightarrow t$.

The first clause of the above definition is directly inherited from strongly convergent infinitary rewriting. The second clause is new. The first part the second clause ensures that the positions in $(q_\beta)_{\beta < \gamma}$ actually track a subterm which is “pulled up” or “pushed down” and that this subterm is an outermost one with this property relative to $(p_\beta)_{\beta < \gamma}$.

The second part of the second clause defines which function symbol occurs at position p . This is either a function symbol from s or it is a symbol that occurs in a subterm that is “pulled up” or “pushed down”, in which case a

sub-reduction of $(t_\beta)_{\beta < \alpha}$ is considered. Note that we might have to “descend” through transfinitely many of such sub-reductions before we encounter the one that is responsible for creating the symbol at position p . It is implicit in the definition that the number of steps in this “descent” is limited by some ordinal.

Example 4.9. It is easy to see that the reductions in (1), (2), and (3) are strongly continuous. Extending the reductions with, respectively, $f^\omega(c)$, c , and $g^\omega(f(h^\omega(c)))$ and employing the pairs of sequences from Example 4.7 we obtain three strongly convergent reductions.

Consider the transfinite term $f(g^\omega(c), g^\omega(d))$ and the rule $f(g(x), g(y)) \rightarrow f(f(x), y)$. We have the following strongly continuous reduction:

$$\begin{aligned} f(g^\omega(c), g^\omega(d)) &\rightarrow g(f(g^\omega(c), g^\omega(d))) \rightarrow \dots \\ &\rightarrow g^n(f(g^\omega(c), g^\omega(d))) \rightarrow g^{n+1}(f(g^\omega(c), g^\omega(d))) \rightarrow \dots \end{aligned}$$

This reduction has two sequences of positions with the outermost pp-property: $((1^\beta)_{\beta < \omega}, (1^\beta \cdot 1 \cdot 1^\omega)_{\beta < \omega})$ and $((1^\beta)_{\beta < \omega}, (1^\beta \cdot 1 \cdot 2^\omega)_{\beta < \omega})$; both are for the term $s = f(x, y)$. Since the number of function symbols above each contracted redexes increases with each step and since these symbols are equal to g , it follows that $g^\omega(f(c, d))$ is the only possible transfinite term that extends the reduction into a strongly convergent one. Note that in the limit we do not have to make a choice between either c or d occurring, as f will also occur.

Now consider the rule $c \rightarrow f(c, c)$ and the reduction

$$\begin{aligned} c &\rightarrow f(c, c) \rightarrow^2 f(f(c, c), f(c, c)) \\ &\rightarrow^4 f(f(f(c, c), f(c, c)), f(f(c, c), f(c, c))) \rightarrow \dots, \end{aligned} \quad (4)$$

which for each finite depth d contracts all $c \rightarrow f(c, c)$ -redexes and then continues to contract all such redexes at depth $d + 1$, &c. Consider the set of positions

$$\{p \mid |p| \leq \omega \text{ and } \forall \beta \in |p| : p(\beta) \in \{1, 2\}\}$$

and define the following map over this set:

$$t(p) = \begin{cases} f & \text{if } |p| \text{ is finite} \\ c & \text{if } |p| = \omega \end{cases}$$

It is easy to see that t defines a term (see Figure 4). Extending the above reduction with t yields a strongly convergent reduction. We need \aleph_1 pairs satisfying the outermost pp-property, one for each c in t . For each pair, $s = c$.

Remark 4.10. Strongly convergent reductions as defined above in general do not satisfy a compression property, i.e. if $s \rightarrow t$, then there might not exist a reduction $s \rightarrow^{\leq \omega} t$. To see this, consider the reduction from (4) extended with the term t defined immediately below it and contract one of the $c \rightarrow f(c, c)$ -redexes in t . This yields a strongly convergent reduction of length $\omega + 1$. This reduction is not compressible, as the contracted redex only comes into existence after ω steps.

The definition of strongly continuous reductions depends on descendants being defined for strictly shorter reductions than reduction we are considering at any one point. Hence, both strongly continuous reductions and descendants need to be defined by mutual transfinite induction. The definition of descendants is as follows:

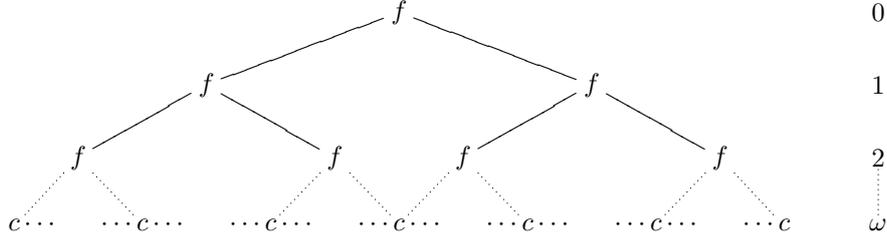


Figure 4: The term t from Example 4.9.

Definition 4.11 (Descendants). Let $t_0 \twoheadrightarrow^\alpha t_\alpha$ and let $P \subseteq \mathcal{Pos}(s_0)$. The set of *descendants* of P across $t_0 \twoheadrightarrow^\alpha t_\alpha$, denoted $P/(t_0 \twoheadrightarrow^\alpha t_\alpha)$, is defined as follows:

- if $\alpha = 0$, then $P/(t_0 \twoheadrightarrow^\alpha t_\alpha) = P$,
- if $\alpha = 1$, then $P/(t_0 \twoheadrightarrow t_1) = \bigcup_{p \in P} p/(t_0 \twoheadrightarrow t_1)$ (see Definition 3.6),
- if $\alpha = \beta + 1$, then $P/(t_0 \twoheadrightarrow^{\beta+1} t_{\beta+1}) = (P/(t_0 \twoheadrightarrow^\beta t_\beta))/(t_\beta \twoheadrightarrow t_{\beta+1})$, and
- if α is a limit ordinal, then $p \in P/(t_0 \twoheadrightarrow^\alpha t_\alpha)$ if there exist a $\kappa < \alpha$ such that for p either the first case from Definition 4.8 applies and $p \in P/(s_0 \twoheadrightarrow^\beta s_\beta)$ for all $\kappa \leq \beta < \alpha$, or the second case applies and there is a sequence $(r_\beta)_{\kappa \leq \beta < \alpha}$ with $r_\beta \in \mathcal{Pos}(t_\beta)$ for all $\kappa \leq \beta < \alpha$ such that:

- $r_\kappa \in P/(t_0 \twoheadrightarrow^\kappa t_\kappa)$, and
- r_β is a descendant of r_γ for all $\kappa \leq \gamma < \beta < \alpha$,

and such that there exists a pair $((p_\beta)_{\kappa \leq \beta < \gamma}, (q_\beta)_{\kappa \leq \beta < \gamma})$ with the outermost pp-property for a term s such that either:

- $(r_\beta)_{\kappa \leq \beta < \alpha} = (p_\beta)_{\kappa \leq \beta < \alpha}$ with $\lim_{\beta \rightarrow \alpha} p_\beta = p$, or
- $(r_\beta)_{\kappa \leq \beta < \alpha} = (q_\beta \cdot q'_\beta)_{\kappa \leq \beta < \alpha}$ for some $(q'_\beta)_{\kappa \leq \beta < \alpha}$ with $p = p' \cdot q' \cdot q''$ and $\lim_{\beta \rightarrow \alpha} p_\beta = p'$, and $q' \in \mathcal{Pos}_X(s)$ such that $q'' \in \{q'_\kappa\}/u$ where u is the reduction obtained by considering $(t_\beta|_{q_\beta})_{\kappa \leq \beta < \alpha}$ and removing all empty steps.

The first three clauses and the first part of the fourth clause are directly inherited from strongly convergent infinitary rewriting [8]. The second part of the fourth clause is new and corresponds to the second clause of Definition 4.8. In the second part a case distinction is made between p being either a position of a function symbol of s immediately below $\lim_{\beta \rightarrow \alpha} p_\beta$ or p occurring at some other position.

In case p is the position of function symbol of s immediately below $\lim_{\beta \rightarrow \alpha} p_\beta$, the definition degenerates to setting p equal to $\lim_{\beta \rightarrow \alpha} p_\beta$. This corresponds to the behaviour of descendants across reduction *steps*, where any position in the redex pattern of a contracted redex descends to the root position of that redex. In case p occurs at some other position, we again consider a sub-reduction.

We have the following:

Lemma 4.12 (Closure under Contexts and Substitutions). *If $(t_\beta)_{\beta < \alpha}$ is a strongly continuous reduction over transfinite terms, then so is $(C[\sigma(t_\beta)])_{\beta < \alpha}$ for arbitrary (transfinite) contexts $C[\square]$ and (transfinite) substitutions σ .*

Proof. Trivial, using Proposition 3.5 and observing that no reductions occur in either the context $C[\square]$ or the terms that are substituted for variables. \square

This ends our discussion of reductions over transfinite terms. Unfortunately, the given definition is mightily complicated and, hence, although we suffered a topological defeat we now only booked what seems to be a Pyrrhic victory.

5 Conclusion

We have defined rewriting over transfinite terms, having first defined transfinite terms and one-step rewriting over these terms. Both the definition of transfinite terms and the definition of one-step rewriting were easy to come by. Unfortunately, the same cannot be said of the definition of strongly convergent reductions. This is especially so since we want to have the ability to “pull up” subterms from infinite depth to finite depth and to “push down” subterms from finite depth to infinite depth.

Although the definition of strongly convergent reductions seems to work quite well for our examples, it is difficult to say—due to its size—whether the definition fully complies with our intuition or whether it is even correct. To gain confidence, it would tremendously help if we could present rigid mathematical evidence, in the form of a number of lemmata, showing that the definition actually achieves what it is supposed to achieve. Unfortunately, we currently do not know what this rigid evidence should be.

Besides the above, we want to point out that at least one reduction with a reasonable limit can be constructed that falls outside the scope of our definition: Consider the rewrite rules $c \rightarrow f(c)$ and $f(x) \rightarrow g(f(x))$. We have the following strongly continuous reduction:

$$c \rightarrow f(c) \rightarrow^2 g(f(f(c))) \rightarrow^3 g^2(f(g(f(f(c)))))) \\ \rightarrow^4 g^3(f(g^2(f(g(f(f(c))))))) \rightarrow \dots$$

This reduction is constructed by iteratively “marking” all $f(x) \rightarrow g(f(x))$ -redexes in a term and contracting all marked copies followed by contracting the $c \rightarrow f(c)$ -redex at the bottom. Intuitively, one would expect the limit of this reduction to be $(g^\omega f)^\omega(c)$, omitting parenthesis to ease notation. In fact it easily follows by the definition of strongly convergent reductions that any possible limit must be of the form $(g^\omega f)^\omega(t)$ for some t . However, that $t = c$ does not follow: By the time we reach the position of c in $(g^\omega f)^\omega(c)$ by “descending” through sub-reductions as in Definition 4.8, the sub-reduction will have length 0; the value of κ employed in the definition approaches ω , as κ depends on the function symbols f that are created along the reduction.

Finally, we have yet to establish if strongly convergent rewriting as defined by us suffices for orthogonal systems to be confluent in the face of strongly convergent reductions, which is of course one of the ultimate aims.

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